Numerical results on photon redistribution and Bose condensation in strongly magnetized plasmas

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We calculate the time evolution of a distribution function for a photon gas interacting with free electrons in a strong magnetic field. Numerical solutions of the kinetic equation are presented including the possibility of a Bose condensation if absorption processes are neglected; the structure of the final equilibrium solution is discussed. Only if the average photon energy is above the cyclotron energy does the photon redistribution show a completely different dependence compared to the nonmagnetic case, due to the resonant scattering process.

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I. INTRODUCTION

The kinetics of a photon gas interacting with free electrons has been investigated by a number of authors with special emphasis on the Bose condensation of the photons in a scattering dominated environment. Zel'dovich, and Levich [1] showed that this condensation process can be calculated in the framework of the Kompaneets equation [2] where the photon-electron scattering is approximated by a diffusion process in energy space. As a result, they obtained an estimate for the finite condensation time which depends on the initial photon spectrum. For large occupation numbers a "shock front" in energy space develops, shifting the excess photons towards lower frequencies where they pile up at the lowest energy state $h\nu=0$. Chapline, Cooper, and Slutz [3] give a numerical solution of the kinetic equation. Coste and Peyraud [4] considered the kinetics of the Bose condensation in some detail, and they particularly discuss the structure of the final stationary photon distribution. In all cases a spatially homogeneous and isotropic system has been assumed where Compton scattering in the nonrelativistic limit is the dominant photon-matter interaction process.

In the present paper we consider the photon kinetics for the case of an external strong magnetic field. Such fields (up to 10^{13} G) occur close to the surface of magnetic neutron stars, and the detection of cyclotron lines in the spectra of x-ray pulsars (first discovered by Trümper *et al.* [5]), has stimulated a large amount of work concerning physical processes in these strong fields including single and double Compton scattering, cyclotron emission, free-free emission, and pair creation. Since the magnetic field specifies a particular direction in space the system is now no longer isotropic. In addition, the cross sections depend on the photon polarization, and become strongly energy dependent showing a resonance structure at the cyclotron energy $h\nu_B=m_ec^2B/B_{\rm cr}$ where $B_{\rm cr}=4.41\times10^{13}$ G is the critical magnetic-field strength. The electrons can move freely along the field lines but occupy discrete energy states (the Landau levels) perpendicular to the field (for details see [6]). It is obvious that the modification of the cross sections with respect to the nonmagnetic case will strongly alter the kinetics of the scattering and condensation process, and we shall calculate the time evolution of the photon-electron system numerically.

II. THE KINETIC EQUATION FOR STRONG MAGNETIC FIELDS

We consider the time evolution of a photon distribution in a spatially homogeneous fully ionized plasma including a strong magnetic field in the z direction. The ions are just needed as background particles for the sake of charge neutrality. To simplify the entire calculation we do not model the evolution of the electrons but assume for all times a one-dimensional Maxwellian distribution along the magnetic field with a constant temperature T_e, and a population of the lowest Landau level only; this requires $kT_e \ll h\nu_B$. The electrons thus act as a heat reservoir for the photon gas. The energy-momentum transfer between the photons and the electrons is dominated by Compton scattering provided the electron density is sufficiently low in order to ignore free-free emission and absorption.

For the ratio of the typical scattering and the free-free absorption time scales we get for $h\nu \ll kT$

$$\frac{t_{\rm sc}}{t_{ff}}=\frac{\sigma_{ff}}{\sigma_T}\approx10^{-4}\left[\frac{n_e}{10^{20}\ {\rm cm}^{-3}}\right]\times\left[\frac{5\ {\rm keV}}{kT_e}\right]^{3/2}\left[\frac{1\ {\rm keV}}{h\nu}\right]^2\bar{g}\ , \quad (1)$$

where $\sigma_T=6.65\times10^{-25}$ cm^2, n_e is the electron density, and $\bar{g}$ is an average Gaunt factor [7]. Considering evolution times for some given initial photon distribution much larger than $t_{\rm sc}$ but shorter than t_{ff}, the system is scattering dominated; however, in the very-low-energy regime this condition will always be violated, and photons will be absorbed and emitted by the bremsstrahlung process.

Cyclotron absorption and emission can be treated as scattering because of the extremely short lifetime of excited Landau states

$$t_{\text{cyc}} \cong 100 \frac{m_e c^2}{h\nu_B^2} \cong 1.3\times 10^{-19} \left[\frac{B_{\text{cr}}}{B} \right]^2 s \; . \tag{2}$$

Taking into account only single Compton scattering the total photon number is conserved, and the kinetic equation can be written as

$$\frac{1}{c}\frac{\partial f}{\partial t} = 2\pi \int_0^\infty d\nu' \int_{-1}^{1} d\mu' [S(\nu,\mu \to \nu',\mu')(1+f')f - \left[\frac{\nu'}{\nu}\right]^2 S(\nu',\mu' \to \nu,\mu) \times (1+f)f'] \; , \tag{3}$$

where $f(t,\nu,\mu)$ is the polarization averaged photon occupation number, $f'=f(t,\nu',\mu')$, and μ denotes the cosine of the propagation angle between the photon direction and the magnetic field, i.e., the z direction. This equation already includes the assumption that f is independent of the azimuthal angle in the x-y plane. $S(\nu,\mu \to \nu',\mu')$ is the differential scattering probability for Compton scattering from (ν,μ) to (ν',μ') averaged over the given electron distribution and over the photon polarization. Obviously Eq. (3) conserves the photon number

$$N = \frac{4\pi}{c^3} \int_0^\infty d\nu \int_{-1}^{+1} d\mu \, \nu^2 f \; . \tag{4}$$

The differential scattering cross section has been calculated by Nagel [8] in the nonrelativistic limit $h\nu \ll m_e c^2$, $kT_e \ll m_e c^2$. It shows a peak at $\nu = \nu'$ with a thermal width given by T_e, and an additional strong cyclotron resonance at a position given by the energy-momentum balance of the scattering process.

Since $S(\nu,\mu \to \nu',\mu')$ has this double-peak structure, i.e., a resonance and a thermal peak, the energy change per scattering $|\nu-\nu'|/\nu$ is not necessarily small for all energies, and it is therefore not possible to approximate the scattering integral in the kinetic equation (3) by a Fokker-Planck–type operator similar to the Kompaneets equation with a diffusion coefficient depending on ν_B and T_e. In the present work we use a slightly modified version of the cross section by starting from the fully relativistic expression derived by Herold [9], and then taking the nonrelativistic limit. This leads to some correction terms in the resonant part of the cross section due to the electron recoil resulting in a shift of the position of the cyclotron resonance peak. A plot of the total cross section used in our numerical calculations is shown in Fig. 1.

It can be shown that $S(\nu,\mu \to \nu',\mu')$ fulfills a useful detailed balance relation

$$S(\nu',\mu' \to \nu,\mu) = \left[\frac{\nu}{\nu'}\right]^2 e^{(h\nu'-h\nu)/kT_e} S(\nu,\mu \to \nu',\mu') \; . \tag{5}$$

Inserting this the kinetic equation (3) reads as

$$\frac{\partial f}{\partial \tau} = \int_0^\infty dx' \int_{-1}^{+1} d\mu \, \hat{S}(x,\mu \to x',\mu') \times [(1+f')f - e^{x'-x}(1+f)f'] \; , \tag{6}$$

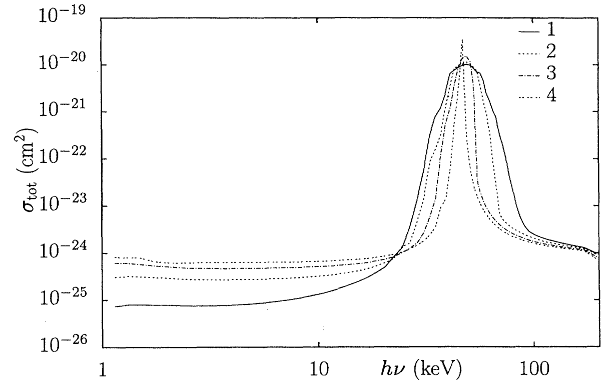

FIG. 1. The total cross section as a function of photon energy used in the numerical calculations. The four curves correspond to different propagation angles with respect to the magnetic field ($1{:}\mu=0.282$; $2{:}\mu=0.618$; $3{:}\mu=0.866$; $4{:}\mu=0.989$). The electron temperature and the cyclotron frequency are 10 and 50 keV, respectively.

where $x = h\nu/(kT_e)$, $\hat{S} = 2\pi kT_e/(hn_e\sigma_T)S$, and τ is measured in units of the Thomson time scale $t_{\text{sc}} = 1/(cn_e\sigma_T)$.

III. THE SOLUTION OF THE KINETIC EQUATION

The kinetic equation (6) is solved numerically from a given initial distribution $f_0(x,\mu) = f(t=0,x,\mu)$ to its final asymptotic solution. In the following we first discuss some properties of this stationary solution, and then present the numerical results in detail.

A. The stationary solution

For the discussion about possible equilibrium states we consider the photon gas enclosed in a large box of finite volume with reflecting walls. Making the linear dimensions of the box sufficiently large we can still assume the energy states to be continuous, however, we use the fact that the lowest possible state (the ground state) now has a finite energy $x_0 > 0$. Then the distribution

$$f(x) = \frac{1}{e^{x-\alpha}-1} \tag{7}$$

is obviously a stationary solution of the kinetic equation (6) under the condition

$$x \geq x_0 > 0; \quad \alpha < x_0 \; , \tag{8}$$

where the chemical potential α is allowed to take a range of positive values. For $\alpha = 0$ $f(x)$ is the occupation number of the usual blackbody spectrum. The total photon number density N reads as

$$N = 8\pi \left[\frac{kT_e}{hc}\right]^3 \int_{x_0}^\infty x^2 f(x) dx = 8\pi \left[\frac{kT_e}{hc}\right]^3 \left\{ \sum_{j=1}^\infty e^{-j(x_0-\alpha)} \left[\frac{x_0^2}{j} + \frac{2x_0}{j^2} + \frac{2}{j^3} \right] \right\} \tag{9}$$

and since the photon number is conserved N is a constant given by the initial distribution. We now consider the transition $x_0 \to 0$. For $\alpha < 0$ this limit is straightforward,

$$N = 16\pi \left[\frac{kT_e}{hc} \right]^3 \sum_{j=1}^{\infty} \frac{e^{j\alpha}}{j^3} \le N(\alpha=0) = 16\pi \left[\frac{kT_e}{hc} \right]^3 \zeta(3) = N_0 , \tag{10}$$

where ζ is the Riemann ζ function. Inverting this yields $\alpha = \alpha(N)$ which completes the description of the stationary solution according to Eq. (7).

For $0 < \alpha < x_0$ the analysis has to be more careful. In the limit of small x_0 we get the Euler summation formula

$$N = 8\pi \left[\frac{kT_e}{hc} \right]^3 \{ x_0^2[-\ln|\eta| + O(1)] + 2x_0[\zeta(2) + O(\eta \ln \eta)] + 2[\zeta(3) + O(\eta)] \} , \tag{11}$$

where $\eta = x_0 - \alpha \ll 1$. From that

$$\alpha(N, x_0) - x_0 \cong -\exp \left\{ - \left[\frac{hc}{kT_e} \right]^3 \frac{N - N_0}{8\pi x_0^2} + \frac{2\zeta(2)}{x_0} + O(1) \right\} \tag{12}$$

which is a good approximation if

$$N - N_0 > \frac{8\pi^3}{3} x_0 (kT_e/hc)^3 \tag{13}$$

and thus covers the case where the initial photon number is larger than $N_0 = N(\alpha=0)$. The distribution (7) then shows a peak close to $x = x_0$ containing photons that have undergone Bose condensation,

$$f(x_0) \cong \exp \left\{ \left[\frac{hc}{kT_e} \right]^3 \frac{N - N_0}{8\pi x_0^2} \right\} , \tag{14}$$

and in the limit ($x_0 \to 0$) $f(x_0)$ diverges whereas the integral (9) over $f(x)$ remains finite. In the regime $x \gg x_0$ $x^2 f(x)$ resembles a Plancklike distribution with $\alpha = 0$.

Note that for the case $N > N_0$ the equilibrium solution is obtained by first solving the kinetic Eq. (3) in the limit $t \to \infty$ for finite values of x_0 and then considering the limiting case $x_0 \to 0$; these two limits are *not* interchangeable. However, taking into account the absorption of photons by the inverse bremsstrahlung process destroys the peak of $f(x)$ at $x = x_0$ in the limit $x_0 \to 0$, and will therefore lead to an equilibrium distribution with $\alpha = 0$.

B. The solution of the time-dependent problem

In order to investigate the time evolution of the photon distribution we start with an initial spectrum peaked around some frequency x_i with a width Δx

$$f(t=0, x, \mu) = \frac{2\zeta(3)}{\sqrt{\pi}} \frac{N}{N_0} \frac{1}{\Delta x x_i^2} \exp \left[- \left[\frac{x - x_i}{\Delta x} \right]^2 \right] \Phi(\mu) \tag{15}$$

where the angle-dependent factor $\Phi(\mu)$ is normalized to one, and for the total photon number density we have (assuming $\Delta x \ll x_i$)

$$4\pi \int_0^{\infty} dx \int_{-1}^{+1} d\mu \, x^2 f(0, x, \mu) \cong \frac{N}{N_0} ; \tag{16}$$

thus, Bose condensation is expected to occur for $N > N_0$. Because the scattering process contains a strong resonance at the cyclotron frequency (see Fig. 1) the overall time evolution will strongly depend on the position of the initial peak with respect to ν_B, i.e., on the ratio x_i / x_B.

Figure 2 shows an example of a numerical calculation with $x_i = 1.95 x_B$, and $N < N_0$, i.e., almost all photons have energies above the cyclotron frequency and above the thermal maximum of the final distribution (note that $h\nu_B \gg kT_e$). In the nonmagnetic case the distribution becomes isotropic much faster than it reaches the equilibrium in energy space; this is a consequence of the nonrelativistic nature of the scattering process. In a strong magnetic field this is not always the case for all energies, since the scattering time scale is drastically reduced around the cyclotron resonance. As a result, the spectrum in this energy range rapidly converges to the equilibrium (7). However, in the high-energy regime the distribution remains anisotropic on a longer time scale (see Fig. 3).

Since the scattering cross section drops off steeply below the Thomson value for low energies, photons will pile up in the wing of the line and then slowly migrate towards the final equilibrium. Therefore, with respect to large time scales the case $x_i > x_B$ is equivalent to a situation with an initially peaked distribution with $x_i < x_B$ and

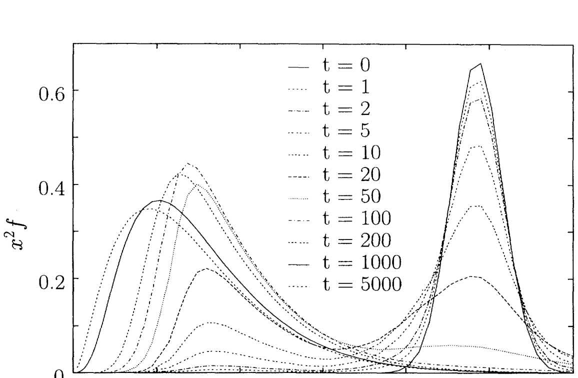

FIG. 2. Time evolution of $x^2 f$ for an initial distribution peaked above the cyclotron energy $h\nu_B = 50$ keV. The electron temperature is 10 keV, and an initial anisotropy factor $\Phi(\mu) = 3/4(1-\mu^2)$ was chosen, all curves are averaged over μ. Different curves correspond to different evolution times in units of t_{sc}. The final stationary solution was obtained at $t \approx 5000 t_{sc}$.

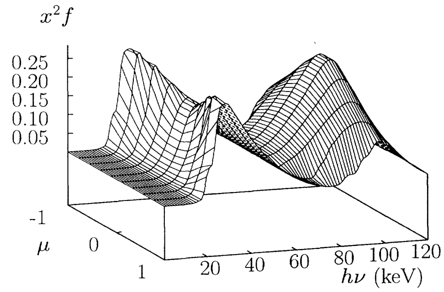

FIG. 3. Frequency and angle dependence of $x^2 f$ at $t=1.8t_{sc}$ for the calculation shown in Fig. 2.

an isotropic angle factor $\Phi(\mu)=\frac{1}{2}$. For $x_i \ll x_B$ the time evolution resembles the nonmagnetic case because the cross section depends only weakly on both energy and direction. Then the scattering integral of the kinetic equation (6) can be simplified by a Kompaneets operator with a diffusion coefficient D depending on the electron temperature and the magnetic field; for $x \ll x_B$

$$x^2 \frac{\partial f}{\partial \tau} = \frac{\partial}{\partial x} \left\{ D(x, x_B, T_e) \left[f(1+f) + \frac{\partial f}{\partial x} \right] \right\} \quad (17)$$

(note that in the nonmagnetic case $D=x^4$). A corresponding numerical example is shown in Fig. 4, where we have chosen $N > N_0$.

The given initial distribution will diffuse through frequency space towards lower energies. First, the bulk of the photons moves into the regime where the stationary state of $x^2 f(x)$ has its maximum, i.e., $h\nu \approx 1.6kT_e$, and then on a large (but finite) time scale the low-energy photons will start condensating. Characterizing the actual spectra in terms of the properties of the usual blackbody radiation we can define a color temperature T_c from the maximum of $x^2 f(x)$,

$$kT_c = \frac{h\nu_{max}}{1.594} , \quad (18)$$

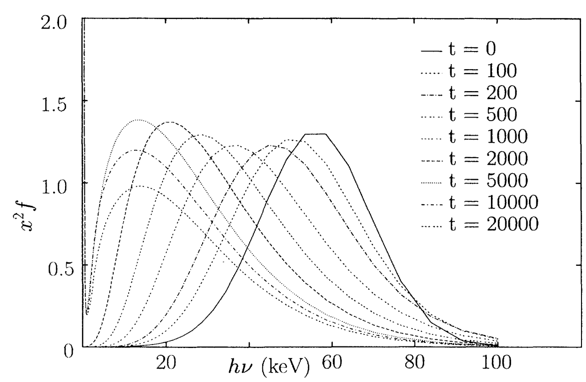

FIG. 4. Time evolution of $x^2 f$ for an isotropic initial distribution peaked below the cyclotron frequency $h\nu_B=200$ keV; the electron temperature is 10 keV. For $t \geq 3000t_{sc}$ photons start condensating into the lowest possible energy scale.

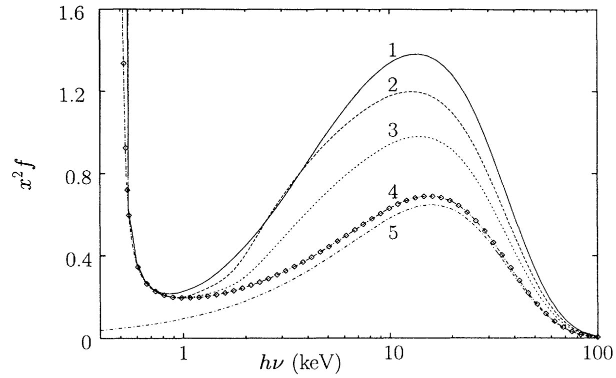

FIG. 5. The final states of the photon redistribution for the calculation shown in Fig. 4. The brightness temperatures at $h\nu=10$ keV for the curves 1,2,3 are $T_b=1.73T_e$, $T_b=1.57T_e$, and $T_b=1.33T_e$, respectively. The lowest-energy state is $h\nu_{min}=x_0 kT_e=0.5$ keV with $kT_e=10$ keV. Curve 4 is a fit of the equilibrium Eq. (7) to the calculated stationary solution (dots) yielding a chemical potential $\alpha=4.99995\times 10^{-2}$ in agreement with Eq. (12). The lowest line is a Bose-Einstein distribution with $\alpha=0$.

as well as a brightness temperature T_b by comparing $f(x)$ with the blackbody brightness at a fixed frequency

$$f(x) = \frac{1}{e^{xT_e/T_b} - 1} . \quad (19)$$

Thus, there is a certain time interval where $T_c \approx T_e$ but $T_b > T_e$ (see Fig. 5), and the photon distribution therefore contains an excess of photons (and energy) with respect to the Bose-Einstein distribution with $\alpha=0$. The final distribution shown in Fig. 5 corresponds to the equilibrium spectrum (7) with the chemical potential from Eq. (12), where the lowest-energy state x_0 is equivalent to the minimum grid point energy $h\nu_{min}=0.5$ keV used in our numerical calculation. Taking into account bremsstrahlung absorption would lead to a blackbody spectrum with $T_c=T_b=T_e$.

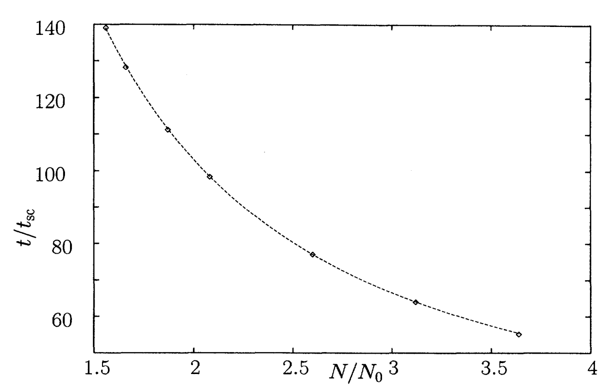

FIG. 6. The condensation time t_C as a function of the initial photon density N/N_0. The dashed line is a fit to the data points $t_C \propto N^{-1.5}$. Electron temperature and cyclotron energy are 10 and 200 keV, respectively.

From an analysis of the Kompaneets equation (17) with an initial distribution (15) Coste and Peyraud [4] obtain for the condensation time t_C

$$t_C \propto N^{-1} \quad (x_i = \mathrm{const}) \; . \tag{20}$$

In the magnetic case, however, we used the complete scattering integral (3) which leads to a slightly modified function $t_C(N)$ (see Fig. 6)

$$t_C \propto N^{-1.5} \quad (x_i \ll x_B) \; , \tag{21}$$

and a typical value of t_C for $N \approx N_0$ is

$$t_C \approx 10^4 t_{\mathrm{sc}} \approx 5\times 10^{-11} \left[\frac{10^{22}\ \mathrm{cm}^{-3}}{n_e} \right] s \; . \tag{22}$$

IV. SUMMARY

The resonant Compton scattering in a strong magnetic field leads to a completely different time evolution of a given photon spectrum compared to the nonmagnetic case if most photons have energies above the cyclotron frequency. Once the average photon energy is below this value this difference becomes unimportant except for a change of time scales. The calculations regarding the process of Bose condensation are based on the assumption of negligible bremsstrahlung absorption in order to keep the photon number constant. Since this condition is never fulfilled for low photon energies the solution for $t \to \infty$ will not contain a condensed phase, but if the absorption rate is small a certain surplus of photons with respect to the blackbody spectrum will show up temporarily, and the magnitude of this excess depends on the initial photon number. This effect is extremely pronounced in the case of low electron densities. On the other hand, once the density is so low that the radiation energy density exceeds the thermal energy density of the plasma the assumption of a constant electron temperature cannot be justified. This condition, however, can in principle be replaced by some other condition like a constant total-energy density $U = U_{\mathrm{pl}} + U_r$ of the plasma-radiation system. If in the nonmagnetic case the radiation is energetically dominant ($U \approx U_r$) the electrons have a Maxwellian distribution where the temperature is adjusted by the Compton recoil on a time scale [10]

$$t_{\mathrm{rec}} = \frac{m_e c}{U_r \sigma_T} \approx 3\times 10^{-11} \left[\frac{10\ \mathrm{keV}}{kT_e} \right]^4 s \; .$$

In a magnetized plasma a similar argument holds if $kT_e \ll h\nu_B$; close to the cyclotron energy the electrons do not necessarily have an equilibrium distribution for all times because Compton scattering can lead to a Landau-level transition with a subsequent emission of a cyclotron photon. The discussion about the stationary solution (Sec. III A) is still valid if T_e is interpreted as the final temperature in the limit $t \to \infty$. As an additional relation one has for the radiation energy

$$U \approx U_r(T_e, \alpha, x_0) = 8\pi hc \left[\frac{kT_e}{hc} \right]^4 \int_{x_0}^{\infty} x^3 f(x)\, dx \; ,$$

and together with $N(T_e, \alpha, x_0)$ the temperature and the chemical potential can be expressed in terms of the constants of the system: $T_e(U_r, N)$ and $\alpha(U_r, N)$ (in the limit $x_0 \to 0$); again, the case $N > N_0$ would lead to a condensation.

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